DIFFERENTIAL EQUATIONS LECTURE NOTES

These notes are intended to supplement sections 6.1 and 4.1 from Nagle, Saff, and Snider. They provide some background and stronger connections to linear algebra which are missing from the text.

INTRODUCTION

An n^{th}**-order differential equation** is an equation in the variables x, y, y', y'', $\ldots$, $y^{(n)}$.

A **solution** to a differential equation is a function $f(x)$ which satisfies the equation when f is plugged in for y and the derivatives of f for y', y'', etc.

Examples:	$y' = \cos(x)$	$y'' = y$	$xy' = y$
Solutions:	$y = \sin(x) + c$	e^t, e^{-t}, others?	cx, others?

Differential equations usually have multiple solutions. Initial value information can help us narrow it down.

Example: $y' = \cos(x), y(0) = 3$.
To solve this initial value problem, we note that $y = \int \cos(x)dx = \sin(x)+c$ describes all solutions to the differential equation. Then $y(0) = \sin(0) + c = c$, so $c = 3$. Hence there is a unique solution: $y = \sin(x) + 3$.

General Principle of Differential Equations: (Existence and Uniqueness of Solutions)
Given a differential equation and "enough" initial value information, there should *exist* a *unique* solution to the initial value problem.

LINEAR DIFFERTIAL EQUATIONS

A n^{th}**-order linear differential equation** sets a linear combination of y, y', $\ldots$, $y^{(n)}$ with coefficients functions of x, equal to a function of x:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

Dividing by a_n (this will restrict us to a domain on which $a_n(x) \neq 0$), we get the **standard form**:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = g(x).$$

The equation is **homogeneous** if $g(x) = 0$.

Theorem (Existence and Uniqueness for Linear Differential Equations)
If $p_1(x)$, $\ldots$, $p_n(x)$ and $g(x)$ are continuous on the interval (a, b), and x_0 is in (a, b), then for any real numbers γ_0, γ_1, $\ldots$, γ_{n-1}, the initial value problem

$$\begin{cases} y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = g(x) \\ y(x_0) = \gamma_0 \\ y'(x_0) = \gamma_1 \\ \vdots \\ y^{(n-1)}(x_0) = \gamma_{n-1} \end{cases}$$

has a unique solution defined on (a, b).

This theorem is proven in more advance courses on differential equations.

Example: $y'' = x^{-3}, y(1) = 1, y'(1) = -\frac{1}{2}$.

Note that this is a second order equation, so we need to know two piece of initial value information, $y(x_0)$ and $y'(x_0)$.

If $y'' = x^{-3}$, then integrating, $y' = \int x^{-3}dx = -\frac{1}{2}x^{-2} + c_1$, and integrating again, $y = \frac{1}{2}x^{-1} + c_1x + c_2$. This describes all solutions to the differential equation.

Using the initial value information, $y'(1) = -\frac{1}{2} \cdot 1 + c_1 = -\frac{1}{2} + c_1$, so $c_1 = 0$.

And $y(1) = \frac{1}{2} \cdot 1 + 0 \cdot 1 + c_2 = \frac{1}{2} + c_2$, so $c_2 = \frac{1}{2}$.

We have found a solution: $y = \frac{1}{2}x^{-1} + \frac{1}{2}$.

But what interval is it defined on? In order to use the theorem to conclude that our solution is unique, we must specify an interval (a, b). It must contain 1 (our x_0), and it must not contain 0 (since our $g(x) = x^{-3}$ is not continuous there).

We may as well take the largest interval satisfying these conditions: $(0, \infty)$. By the theorem, $y = \frac{1}{2}x^{-1} + \frac{1}{2}$ is the unique solution to the initial value problem on $(0, \infty)$.

As an example of why we need to be careful about the domain interval, consider the function

$$f(x) = \begin{cases} \frac{1}{2}x^{-1} + \frac{1}{2} & x > 0 \\ \frac{1}{2}x^{-1} - 1 & x < 0 \end{cases}$$

which is defined on $\mathbb{R} \setminus \{0\}$. At every point other than 0, f satisfies $f''(x) = x^{-3}$, and we have $f(1) = 1$ and $f'(1) = -\frac{1}{2}$. That is, f is a solution to the initial value problem. The moral is that uniqueness of the solution may fail if we consider functions defined on larger domains.

Example: $xy'' + \sqrt{x+1}y' + \frac{1}{x-1}y = 0, y(x_0) = 0, y'(x_0) = 0$.

Standard form:

$$y'' + \frac{\sqrt{x+1}}{x}y' + \frac{1}{x(x-1)}y = 0.$$

All coefficient functions are defined and continuous on $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. So the theorem guarantees a unique solution defined on

$$\begin{cases} (-1, 0) & \text{if } -1 < x_0 < 0 \\ (0, 1) & \text{if } 0 < x_0 < 1 \\ (1, \infty) & \text{if } 1 < x_0 \end{cases}.$$

Note that if we were given initial value information at three points, one in each interval, then we would get a unique solution on each interval, which we could piece together into a unique solution on the union of the domains.

LINEAR ALGEBRA PERSPECTIVE

We are working in a large vector space of functions. This could be

- C^∞, infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$, or
- $C^\infty(a, b)$, infinitely differentiable functions $(a, b) \to \mathbb{R}$, or maybe
- $C^n(a, b)$, n times differentiable functions $(a, b) \to \mathbb{R}$, if we are working with at most n^{th}-order differential equations.

For convenience, I will usually use $C^\infty(a, b)$ to denote our function space.

Given an n^{th}-order linear homogeneous differential equation $y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=0$, we may pick an interval (a,b), on which $p_1,\ldots,p_n$ are defined and continuous. The differential equation then defines a linear transformation $L : C^\infty(a,b) \to C^\infty(a,b)$, given by

$$L(f) = f^{(n)} + p_1 f^{(n-1)} + \ldots p_n f.$$

The solutions to the differential equations are the functions f in $C^\infty(a,b)$ such that $L(f)=0$, that is, the **kernel** of L. Recall that the kernel of a linear transformation is always a subspace of the domain, so $\ker(L)$ is a subspace of $C^\infty(a,b)$. We are interested in describing this subspace, which I will call $S=\ker(L)$.

For any point x_0 in (a,b) and f in S, let

$$\text{Eval}_{x_0}(f) = \begin{pmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ f^{(n-1)}(x_0) \end{pmatrix}.$$

This is a vector of n real numbers. Eval_{x_0} is a linear transformation $S \to \mathbb{R}^n$.

Now a set of initial values $y(x_0)=\gamma_0,\ \ldots,\ y^{(n-1)}(x_0)=\gamma_{n-1}$ is just a target vector in $\mathbb{R}^n$:

$$\overline{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{pmatrix}.$$

That is, we require $\text{Eval}_{x_0}(y)=\overline{\gamma}$. The existence and uniqueness theorem says that for *any* target vector $\overline{\gamma}$ in $\mathbb{R}^n$, there *exists* a *unique* f in S such that $\text{Eval}_{x_0}(f)=\overline{\gamma}$. The existence says that Eval_{x_0} is *onto*. The uniqueness says that Eval_{x_0} is *one-to-one*.

So Eval_{x_0} is an **isomorphism** $S \cong \mathbb{R}^n$. This tells us several things:

- $\dim(S)=n$.
- If we can find n linearly independent solution functions $f_1,\ldots,f_n$ in S, they will be a basis for S. That is, all solutions to the differential equation can be uniquely expressed as $c_1f_1+\ldots c_nf_n$ for some real numbers $c_1,\ldots,c_n$.
- Given initial value information at x_0, $y(x_0)=\gamma_0,\ \ldots,\ y^{(n-1)}(x_0)=\gamma_{n-1}$, the unique solution to the initial value problem will be $c_1f_1+\cdots+c_nf_n$, where the c's are chosen so that $c_1\text{Eval}_{x_0}(f_1)+\ldots c_n\text{Eval}_{x_1}(f_n)=\overline{\gamma}$.
- Given functions $f_1,\ldots,f_n$ in S (that is, solutions to the same differential equation), we can test their linear independence by testing the linear independence of $\text{Eval}_{x_0}(f_1),\ldots,\text{Eval}_{x_0}(f_n)$. One way to do this is by taking the determinant

$$W[f_1,\ldots,f_n](x_0) = \begin{vmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{vmatrix}.$$

  This determinant is called the Wronskian of $f_1,\ldots,f_n$, evaluated at x_0.